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Nonlinear multimode photonics: nonlinear optics with many degrees of freedom

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The overall goal of photonics research is to understand and control light in new and richer ways to facilitate new and richer applications. Many major developments to this end have relied on nonlinear optical techniques, such as lasing, mode-locking, and parametric downconversion, to enable applications based on the interactions of coherent light with matter. These processes often involve nonlinear interactions between photonic and material degrees of freedom spanning multiple spatiotemporal scales. While great progress has been made with relatively simple optimizations, such as maximizing single-mode coherence or peak intensity alone, the ultimate achievement of coherent light engineering is complete, multidimensional control of light–light and light–matter interactions through tailored construction of complex optical fields and systems that exploit all of light's degrees of freedom. This capability is now within sight, due to advances in telecommunications, computing, algorithms, and modeling. Control of highly multimode optical fields and processes also facilitates quantitative and qualitative advances in optical imaging, sensing, communication, and information processing since these applications directly depend on our ability to detect, encode, and manipulate information in as many optical degrees of freedom as possible. Today, these applications are increasingly being enhanced or enabled by both multimode engineering and nonlinearity. Here, we provide a brief overview of multimode nonlinear photonics, focusing primarily on spatiotemporal nonlinear wave propagation and, in particular, on promising future directions and routes to applications. We conclude with an overview of emerging processes and methodologies that will enable complex, coherent nonlinear photonic devices with many degrees of freedom.

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1. INTRODUCTION

Future advances enabled by nonlinear optics will probably be increasingly due to devices whose underlying physics and design involve many degrees of freedom (DOFs), i.e., many photonic DOFs, such as spatial modes, frequencies, and polarizations, and many design parameters, such as the space- and time-dependent distributions of refractive index and loss/gain in photonic structures. This is an easy prediction to make: as photonics researchers, we are part of a tradition that has always sought to control light in more, and more intricate, ways. Increasingly urgent open questions for nonlinear optical science and engineering, thus, include: When and how can nonlinear photonic devices benefit from more modes? What new capabilities and applications await discovery and invention in the domain of nonlinear optical physics and design with many DOFs?

Control of light with many spatial modes, so-called structured light, has advanced over the past decade, driving progress in optical particle manipulation and imaging [1–3]. Manipulating light in time and frequency domains is also mature and is the domain in which nonlinear techniques, such as mode-locking [4–12] and supercontinuum generation [13–17], have so far played the most crucial role. In comparison, spatiotemporal light control has lagged behind. Why? One reason for this is that simulation of spatiotemporal nonlinear wave propagation is much harder than purely spatial ($2+1$-dimensional) or purely temporal ($1+1$-dimensional) wave propagation; simulation complexity generally scales exponentially with the number of dimensions. A second reason is that conceptual tools, like modes or solitons, and experimental tools, like pulse/wavefront shaping, are more challenging to apply in general spatiotemporal wave settings. Nonetheless, initial steps ranging from extreme-intensity laser accelerators [18–21] to sources of entangled states [22–29] show that motivation and progress are present here too.

Here we present an overview of emerging applications and platforms for *multimode nonlinear photonics*, as well as emerging

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methodologies to address the scientific and engineering challenge of multimode nonlinear optical design. Although our scope is multimode nonlinear photonics generally, this paper places an emphasis on spatiotemporal physics and devices because these represent the ultimate control over light, because they remain immature compared to control in pure-time or pure-space domains, and because this is the domain closest to the authors' collective expertise. Our perspective here is meant to be primarily an engineering-oriented one, since the physics of multimode and spatiotemporal nonlinear optical systems have been reviewed elsewhere [30–33], and because we believe the time is right to begin developing applications, both by judicious engineering based on existing laboratory platforms and by development of new, engineering-friendly platforms for applications of multimode nonlinear optical physics.

The era of multimode photonics has already begun.

Although the development of nonlinear photonics has been driven by many causes, here we consider several representative slices of the field's history—optical telecommunications, imaging, and spectroscopy, and their coupling to advanced nonlinear photonic light sources (Fig. 1). In the past, single-mode structures like fibers and cavities enabled light sources with unprecedented coherence and peak intensity, facilitating new applications. Now, the trend is toward systems that utilize richer, multimode forms of coherent light to improve on applications of single-mode light, as well as to facilitate qualitatively new capabilities and applications.

The terminology "mode" has a varied usage throughout photonics literature [65] and is even more challenging to specify in nonlinear and strongly dissipative settings. For the purposes of this article, we can loosely define "modes" as distinct dimensions or DOF of light, usually in the form of spatial or spatiotemporal eigenfunctions of a relevant linearized system description.

Optical telecommunications has driven the development and refinement of many optical devices—fibers, lasers, amplifiers, modulators, filters, and measurement devices—that have enabled a much wider range of research and smaller-market applications. Nonlinear fiber optics [66] and fiber lasers [67,68], as well as their applications in imaging or manufacturing, are two well-known examples, but nearly every experimental photonics paper published today uses some equipment that was developed or enhanced for optical telecommunications. Trends in optical telecommunications, thus, foreshadow the future of nonlinear photonic devices. Today, optical communications is increasingly pressed against the limits of single-mode devices. State-of-the-art research systems now overwhelmingly rely on multimode optics to increase information capacity [34,69,70]. It is difficult to predict exactly when and how multimode telecommunications will impact commercial systems. Nonetheless, it is not difficult to expect that multimode nonlinear photonic devices will benefit from these advances, regardless of how and when multimode optical telecommunications reaches its tipping point.

Optical imaging and spectroscopy are ancient optical applications that have expanded and refined as new light sources have become available. Initially, both relied on incoherent light. While incoherent versions remain important, these techniques were enhanced by the introduction of coherent single-mode laser light, and then further by forms of multimode coherent light, as in mode-locked lasers. To push beyond the limits of linear imaging and spectroscopy, new techniques increasingly capitalize on high-intensity or quantum-correlated light [35,36]. Major advances have arisen from structured coherent light in space, as in stimulated-emission-depletion (STED) microscopy [37], structured illumination microscopy [38,39], and light-sheet imaging [40], or in time, as in multiphoton [41] or stimulated Raman imaging [42,43]. Whereas trends in light source development have recently followed telecommunication developments, the trend with imaging and spectroscopy has been more diverse. The laser and stabilized frequency comb were developed with spectroscopic motivations [44], and early laser workers envisioned many other applications, such as in surgery [45]. But most techniques enabled by nonlinear optical light sources were not anticipated by their inventors and were instead invented by others, many years after the requisite light source capabilities were first introduced.

If properly designed and controlled, devices and processes based on multimode optical physics offer significant advantages over single-mode approaches.

The motivation for harnessing multimode coherent light is now fairly obvious: In concrete situations we know of, process or device performance is expected to grow linearly, and perhaps superlinearly, with the number of controlled modes. In

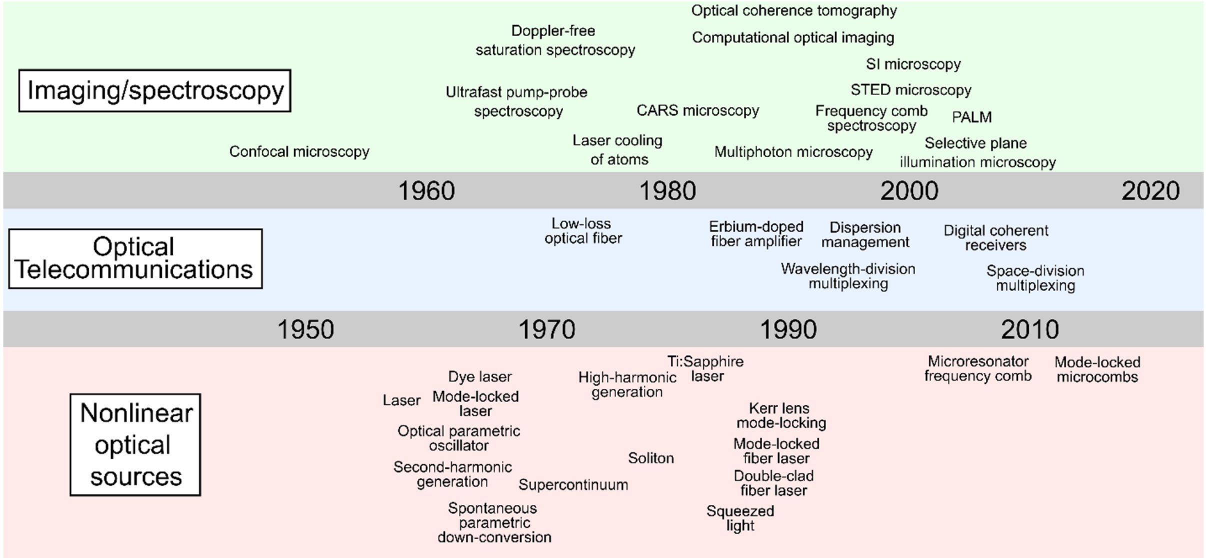


Fig. 1. Timeline of major developments in optical imaging and spectroscopy, optical telecommunications, and nonlinear optical light sources [34–64].

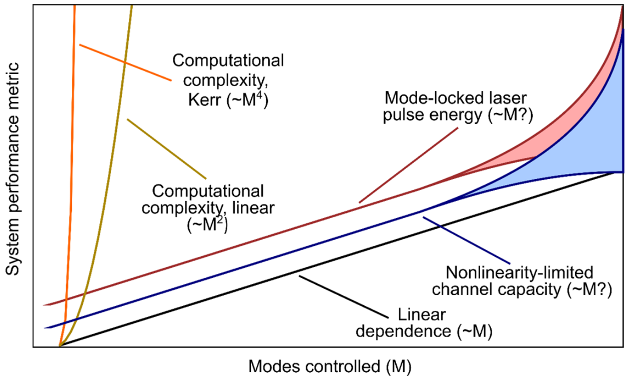


Fig. 2. Scaling of various photonic system performance metrics with the number of controlled modes (M). Communication capacity and laser pulse energy nominally scale linearly with modes, while the computational complexity—an upper bound for the useful computational capacity—scales more rapidly. For some metrics, research has yet to establish large-M scaling, although encouraging signs suggest superlinear scaling for both channel capacity and stable mode-locked laser pulse energy.

optical telecommunications, increasing the number of spatial modes in a fiber channel leads to a nominally linear increase in the Kerr-nonlinearity-limited Shannon capacity of the fiber channel, assuming the modal propagation is properly controlled [34,69,70]. Without increasing the size of the fiber cable, an improvement of 1000 or more is possible. However, once the multimode physics of the waveguide is considered, it turns out that the intrinsic disordered mode coupling that occurs in multimode waveguides may suppress the rate of Kerr-induced cross talk between transmission channels relative to an equal number of single-mode channels [71,72]. In effect, the scaling of capacity with the number of modes controlled may turn out to be even greater than linear (Fig. 2). Similar considerations apply to mode-locked fiber lasers or amplifiers, where the Kerr nonlinearity sets a similar bound on achievable pulse energy [73–75]. Here, it appears that different intracavity pulse evolutions [75] and multimode mode-locking mechanisms exist that can fundamentally tolerate greater amounts of nonlinearity—allowing for similar superlinear scaling with respect to number of modes controlled [73–75]. As an additional example, linear optical computers effectively perform matrix-vector multiplication, a computation whose complexity scales quadratically with the number of dimensions [76,77]. In optical computing devices that exploit nonlinear optical operations, the computational complexity could scale cubically or quartically [78], and in quantum information processing the scaling may even be exponential. In general, the more modes, properly controlled, the better.

What do we mean by "properly controlled"? In general, multimode light is properly controlled if it has *controlled intermodal phases*: the relative phase relationship of all modes should be fixed or controlled over time, not varying stochastically. While incoherent light, which is characterized by stochastic, time-varying intermodal phases, can be useful, e.g., for suppressing speckle artifacts in imaging, most new applications of multimode light require precise control of intermodal phases. Even in settings where spatial disorder is fundamental, such as wavefront shaping through complex media [79,80], or using strong fiber disorder to reduce multiple-input multiple-output (MIMO) processing complexity [81,82], stably controlling the phase relationship between modes is a necessity.

Overall, controlling light in increasingly more modes is perhaps the most obvious way to fundamentally advance the capabilities of light and light-based devices, and its relevance to enhancing and enabling new applications is increasingly diverse. As the intertwined history of nonlinear optical light sources and measurements suggests, many of the most important applications are likely to be discovered and developed in the future. Nonetheless, among the wide range of proposed and existing applications, we can identify several overarching themes regarding where multimode coherent light is desirable for nonlinear optical processes:

- Spatiotemporal wave vector matching in nonlinear interactions, as in nonlinear photonics in bulk or multimode systems [83–91] or control of the spatiotemporal characteristics of spontaneous nonlinear processes, as in virtually all non-classical light generation [22,24–27,29,92–96].
- Control of light–matter interactions or light–light interactions involving very different time and spatial scales, such as in laser materials processing [97–101], the generation of THz or x ray radiation from optical pulses [83–85,88], electronic or plasma [18,19,21,102–110], and phononic interactions [111] or nonlinear microscopy [99,112–114]. These considerations are even more important for strong-field processes.
- Information capacity or complex functionality, as in photonic communication [34,69,70] and information processing [76–78,115,116], as well as for computational and/or high-throughput measurement modalities.

We have hope for controlling multimode optical systems now due to several key developments over the last decade. Well-controlled multimode photonics requires photonic designs and control systems with many DOFs, such as high-resolution optoelectronics or other reconfigurable substrates. These in turn usually require simulations with many-variable models, which require computing power and time. These requirements are even more stringent for well-controlled multimode *nonlinear* photonics because nonlinear systems are more costly to accurately simulate.

Figure 3 shows why these requirements can now be met: the growing accessibility of computing power and of high-resolution, high-performance optoelectronic interfaces, such as cameras, displays, and spatial light modulators (SLMs). Reconfigurable control of many modal DOFs can be achieved with SLMs or digital micromirror devices, which are now available with resolutions (i.e., number of pixels) close to 10^7, or in fixed or reconfigurable means within nanofabricated photonic structures, such as those realized with the silicon photonics platform. In general, the number of design DOFs routinely available to optical scientists and engineers now exceeds by 3 to 5 orders of magnitude the technology of decades ago. This trend is likely to continue in coming decades. Figure 3(b) shows an important driving force for this trend: the ongoing transition of optical telecommunications from single- to multiple-spatial mode, space-division multiplexed (SDM) communication systems. Although the commercialization of such multimode photonic technologies will take years, the economic drive to develop low-cost, high-performance reconfigurable multimode photonics is already present.

We believe the last telling trend for the future of complex, nonlinear photonic system design is the tremendous and continuous

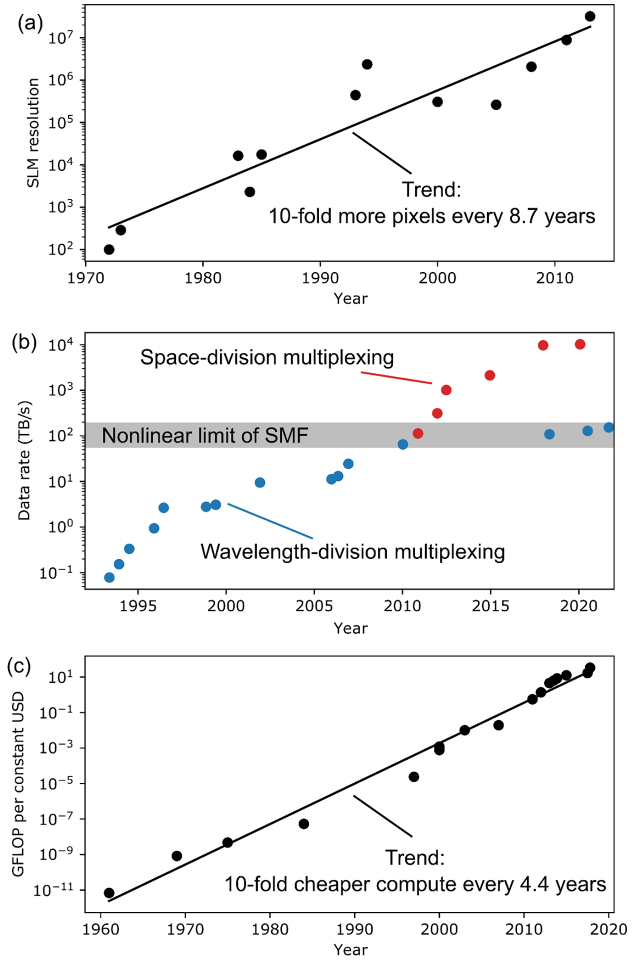


Fig. 3. Progress in computing and photonics. (a) Progress in reconfigurable photonic devices (SLM resolution, or number of pixels, data from [117–130]). (b) State-of-the-art laboratory demonstrations in optical telecommunications (total data rate of the optical channel, data from [70]). (c) Cost of computing (giga-floating-point-operations per USD, data from [131]). Our intention in (a) is to show a representative sampling of devices over time, so points include a variety of diverse SLM types, including both liquid-crystal-based and digital micromirror devices. SLM resolution grows by roughly a factor of 10 every 9 years, while GFLOP-per-constant-USD grows by 10 times every 4.2 years. The growth of telecommunications data rates has slowed, especially if multimode demonstrations are excluded, since systems are now operating very close to the single-mode Kerr-nonlinearity-limited Shannon capacity [34,69,70]. Since high-degree-of-freedom nonlinear photonics has usually been directly limited by the scale of computations that can be performed, it is poised to acutely benefit from ongoing improvements in computing costs.

progress in the cost of computation [Fig. 3(c)]. While some intuitive or analytical methods are emerging for design and prediction of highly multimode, spatiotemporal nonlinear optical systems (see Section 3), the tasks of predicting, designing, and controlling multimode, high-DOF nonlinear photonic systems are among the most computationally intensive endeavors in modern photonics. Since the availability and cost of computation has as a result often directly limiting what is possible in this area of photonics [32], we expect major progress in the coming years simply by leveraging new computational resources. This trend should motivate researchers to continue to explore and develop a wider range of computationally intensive techniques for high-DOF nonlinear photonics, to enable the understanding of new physics, to produce new designs, and to automate complex experiments and devices [132].

For experimental control, such techniques include heuristic optimization algorithms like simulated annealing or genetic algorithms [133], data-driven neural network models and policies [134–139], and physics-informed machine learning models [140–143]. Methods developed for linear wave systems, such as transmission matrices [144–146], or concepts such as principal modes [147–151] and deformation eigenmodes [142], may also be generalizable to nonlinear systems [75,152]. For device design (and to a lesser extent, control), gradient-based optimization techniques, such as gradient descent using autodifferentiation (also known as differential programming) and the adjoint method [78,153–157], apply. These methods have proven surprisingly effective for optical device design, often defying human intuition. Through a combination of efficient algorithms and lowering costs of computing, we anticipate computer-assisted design of nonlinear photonic systems with ever-more DOFs. Deep learning is, in a nutshell, efficient gradient-based optimization of nonlinear functions with many DOFs. The unexpectedly favorable scaling of this paradigm with the number of adjustable DOFs has led to revolutionary progress in automated algorithms over the last decade [158]. A similar evolution appears likely for nonlinear photonics with many DOFs.

2. EMERGING PLATFORMS AND APPLICATIONS

A. Multimode Frequency Conversion in Optical Fibers

From the earliest days of nonlinear fiber optics, multimode fibers have enabled remarkable frequency conversion capabilities, particularly for processes that involve large (10–100 THz) frequency shifts [159–162]. Large frequency differences between waves imply disparate propagation constants and velocities, along with possible multimode propagation of some of the participating waves. Multimode propagation creates challenges such as temporal walk-off for short pulses along with opportunities for phase-matching that build on concepts that have been extensively investigated in integrated waveguides. Frequency conversions of radiation from the continuous-wave to femtosecond domains, and from photon pairs to megawatt peak powers, have been demonstrated in multimode fibers [30]. Intermodal four-wave mixing (FWM) processes including the geometric parametric instability and spatiotemporal dispersive wave generation provide ways to generate spectral sidebands with large offsets [163–169] with either spatiotemporal phase-matching (i.e., chromatic dispersion is compensated by modal dispersion) or quasi-phase-matching through self-imaging propagation in graded-index (GRIN) fibers [170]. The use of radially symmetric higher-order LP_{0n} modes of step-index fiber allows long interactions of large-area modes, with anomalous dispersion [171] and enhanced phase-matching bandwidth for intermodal FWM [172]. Anomalous dispersion supports formation of Raman solitons, which underlie continuous frequency shifting over broad ranges in the near-infrared [173–175] and mid-infrared [176], and Raman beam-cleaning [177] may allow effective brightness-conversion in wavelength-shifting processes [178]. The impressive results described above are sometimes accompanied by complexity in design and implementation, and historically these have posed barriers to adoption of multimode techniques. Optimization of performance (such as spectral concentration of power) for specific applications may increase the motivation to develop practical versions, which in some cases will require new perspectives.

One area where multimode nonlinear processes may find near-term application is telecommunications. FWM is the basis of several optical signal processing functions, and intermodal processes can be exploited to realize those functions in SDM systems. Following the observation that phase-matching of wavelengths several THz apart can be maintained over several km of multimode fiber [179], researchers have demonstrated selective conversion of wavelengths corresponding to different channels [180], as well efficient wavelength conversion with telecom-modulated signals while avoiding excessive noise arising from nonlinearity [181].

B. Multimode Continuum Generation

Continuum generation in multimode fiber complements the capabilities of the now-common continua from single-mode fibers. Multimode continuum generation is extremely complex, with contributions from the phase-matched multi-wave mixing processes [182] mentioned in the prior section along with phase modulations and the production of solitons and dispersive waves in multiple transverse modes. Nevertheless, continua generated in multimode fiber can be controlled to some extent [133,183,184], and adequate beam quality for some applications can be achieved over wide spectral ranges [164,182,185]. Larger mode areas and the presence of modal dispersion in multimode fibers underlie scaling to higher pulse energy and power [186,187], but work remains to control the spectral coherence and temporal profile of slices of the continuum. Non-silica multimode fibers allow extension of continua to mid-infrared wavelengths [188–190]. Further advances in control of the spatial and temporal profiles will build on detailed comparison of theoretical and experimental results, as in [190], and will facilitate tailoring of continuum sources to applications.

C. Multimode Propagation in Hollow-Core Fibers

Gas-filled hollow-core fibers (HCFs) offer major opportunities for exploration of multimode nonlinear wave propagation and engineering of high-performance sources of ultrashort pulses. Whether microstructure fiber or capillary, HCFs combine low modal dispersion with pressure-tunable chromatic dispersion and nonlinearity [191]. Multimode propagation effects have been invoked to explain aspects of frequency conversion ranging from the near-infrared to the vacuum-ultraviolet [192–195], compression of pulses to few-cycle duration accompanied by strong redshifting [196,197] or blueshifting [198], and the observation of new spatiotemporal wave packets [199]. High-harmonic generation (HHG) of ultraviolet light and x rays is an application that can benefit from multimode propagation for phase-matching [85,200–203] as well as from spatiotemporally localized fields [84] that form as a result of excitation of higher-order modes by plasma defocusing [204]. Direct driving of HHG by the wave packets reported in [199] offers exciting potential for practical extreme ultraviolet sources. Progress in engineering new sources across the spectrum will build on controlled excitation of higher-order modes of HCFs [205–207], will require better understanding of highly multimode propagation, and will benefit from the development of HCFs with lower losses for the higher-order modes [208].

D. Multimode Short-Pulse Fiber Amplifiers

It will be interesting to look for nonlinear pulse evolutions in multimode fiber amplifiers that yield desirable spatiotemporal profiles. In dissipative systems, beam-cleaning processes can improve beam quality as measured by M^2 and, thus, facilitate effective scaling of energy or power. Beam-cleaning accompanies amplification of sub-nanosecond pulses in doped GRIN fiber [209] and continuum generation in tapered doped GRIN fiber [210]. Wavefront shaping should be a useful tool for mitigating adverse spatial and temporal consequences of multimode propagation (and disorder), which should allow scaling of amplifier performance and generation of spatiotemporally controlled outputs [211]. Careful measurements of the spatiotemporal pulse evolution along with detailed theoretical modeling will be needed to fully exploit the potential of multimode fibers in short-pulse amplification. Extension of the concepts of principal modes [147] and deformation principal modes [142] to dissipative and nonlinear systems could be valuable for mitigation and possible exploitation of effects of mode coupling and disorder.

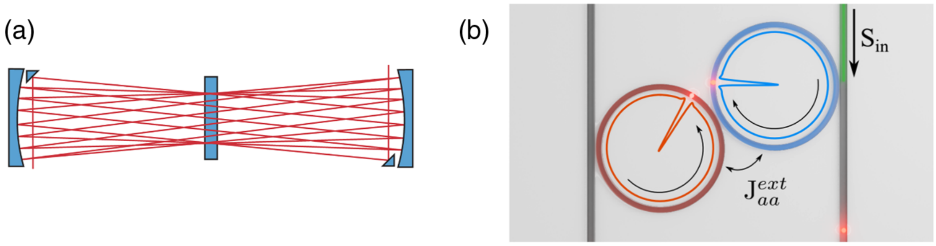
E. Multimode Short-Pulse Fiber Lasers

Application of multimode engineering to fiber lasers could permit low-cost femtosecond lasers (and regenerative amplifiers) with dramatically higher pulse energy and average power, or that generate high-contrast customized spatiotemporal shapes. However, the high-dimensional dynamics that dictate how mode-locking occurs in these systems is challenging to predict. General insights such as the importance of low modal dispersion (e.g., through GRIN or multicore fiber designs) and spatial filtering to compensate modal dispersion enabled the first observations of spatiotemporal mode-locking (STML). [74,212–214]. Key mechanisms behind diverse forms of STML have been identified by “dissecting” a full laser model into nonlinear attractors [75], and numerical studies reveal a variety of spatiotemporal soliton, breather, and vortex solutions [215]. The practical impact of STML will likely be limited until ways to generate high-quality or intentionally structured beams are identified. Kerr beam-cleaning inside an STML laser can improve beam quality [216], and simulations provide glimpses of mode-locked states that yield high-power and near-Gaussian output beams [75]. While the occurrence of STML even in the presence of a significant manufacturing disorder is scientifically remarkable, for some applications disorder will need to be mitigated via intra-cavity wavefront shaping [217] or by novel fiber/cavity designs. On the other hand, disorder may present scientific opportunities such as investigation of lasing based on analogs of principal modes. A holy grail of multimode nonlinear engineering would be a design methodology that allows engineers to precisely control the dissipative, nonlinear self-organization that underlies STML to directly realize pulses with prescribed spatiotemporal features.

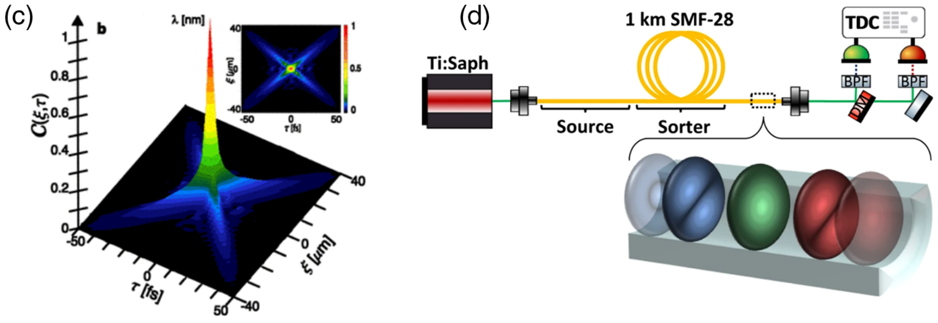
F. Multimode Solid-State Lasers and Cavities

Major advances are occurring in engineering of multimode lasers [218], perhaps best exemplified by the demonstration of control of 300,000 transverse modes, allowing independent and high-resolution manipulation of the output structure in the near and far fields [219]. Interactions of the longitudinal and transverse cavity modes impact the spatial coherence of the output, which can be exploited for speckle suppression in imaging applications

Classical light sources



Quantum light sources



Other potential applications

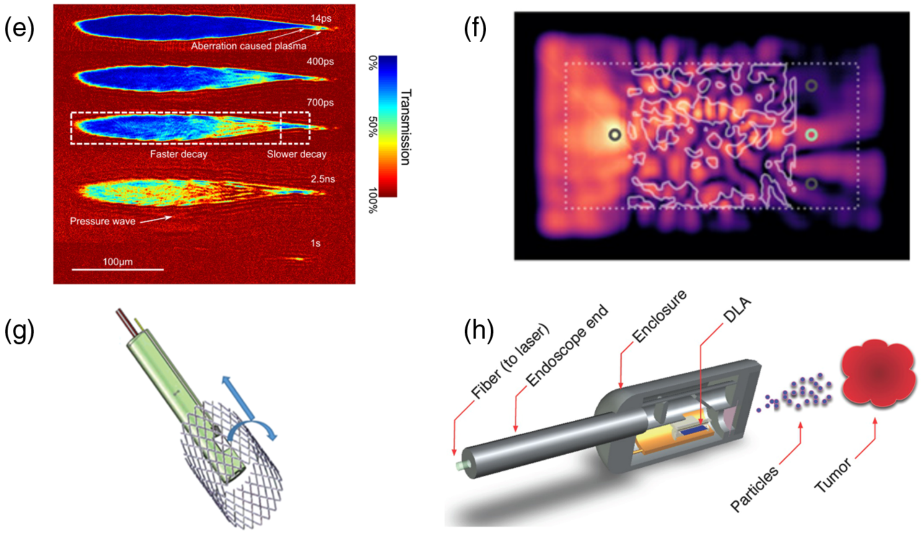


Fig. 4. Platforms and potential applications of multimode nonlinear photonics. (a) Multipass cells [234] and (b) integrated photonic structures [235] are flexible platforms wherein nonlinear multimode waves may be controlled to produce new forms of multimode coherent light. Spatiotemporal and multimode considerations are necessary to describe many spontaneous nonlinear processes for generating non-classical light, e.g., (c) in bulk media [92], or to engineer (d) fiber-based [236] or integrated [93] sources of non-classical multimode light. Applications that may benefit from coherent multimode light include the (e) intrinsically spatiotemporal interactions in laser materials processing and laser-plasma interactions [100]. Multimode nonlinear optical dynamics can be used for information processing, such as all-optical phase-retrieval [237], or (f) machine learning inference [115,238]. (g) Photoacoustic [239] and other biomedical optical imaging modalities, as well as (h) integrated laser-driven particle accelerators [110], may benefit from multimode fiber endoscope-based deployment [240–244]. Figures adapted from [92,100,110,234–236,238,245] with permission.

[220]. Modern short-pulse lasers now routinely operate in intensity regimes in which spatiotemporal nonlinear processes such as self-focusing are relevant [87,221–225]. Although self-focusing is often mitigated simply by larger beam dimensions, it plays an essential role in Kerr lens mode-locking [226,227], as well as bulk continuum generation [228–233]. Similar considerations apply, e.g., to multipass cell compressors (Fig. 4) and bulk multipass amplifiers [246,247], where spatiotemporal and self-focusing effects are typically avoided.

In all the systems above, the modes are perturbed modes of a "cold" cavity. Laser cavity solitons balance diffraction and self-focusing processes in otherwise unstable cavities [248]. Although most work in this area is in the spatial domain, spatiotemporal versions have been investigated [249,250]. Experiments are just scratching the surface of this area, with numerous theoretical investigations of systems that can support self-localized cavity "light bullets" and related wave packets [251–255]. Although multidimensional nonlinear optical waves have historically been harder to control than their low-dimensional counterparts, experimental control of more complex DOFs that have no analog in lower dimensions, such as angular momentum and vortex structures, are increasingly being recognized as promising tools for reliable control of multidimensional nonlinear optical waves [31,239,256]. We think these and other innovations are worth pursuing: In single-mode fiber lasers, exploiting rather than avoiding nonlinear processes has boosted performance by over 3 orders of magnitude [73]; similar success with bulk ultrafast laser systems could enable compact, portable, and low-cost terawatt-class oscillators and petawatt-class master-oscillator power-amplifier systems.

G. Multimode Bulk Nonlinear Photonic Devices

In high-intensity laser applications such as particle acceleration [18,19,21,104–106,110], light–plasma or light–electron interactions [107–109], and micromachining [97,100,101], careful engineering of the spatiotemporal structure of the driving laser pulses is required [257–259]. It is increasingly being appreciated that more complex spatial and spatiotemporal fields can lead to better performance or control in these processes than simple, unstructured Gaussian pulses. This is not surprising given that these processes typically involve a multitude of spatial and temporal scales [18,19,109,258]. While this endeavor is relatively new, we are optimistic that spatiotemporally tailored, high-intensity light sources will facilitate genuine breakthroughs both for enabling low-cost, compact devices with synchrotron-like capacities, as well as for pushing extreme light–matter interactions into qualitatively new regimes.

H. Multimode Integrated Microresonator Frequency Combs

Microresonator frequency combs have attracted enormous interest for the efficient generation of stable octave-spanning frequency combs on an integrated semiconductor chip [260–263]. Broad bandwidths can be generated through Kerr-mediated parametric frequency generation in high quality factor microresonators with low (<1 mW) drive powers. While many early combs featured low coherence [264] with phase variations between the frequency modes, the breakthrough observation of optical solitons enabled highly coherent mode-locked frequency combs [12,63,265]. Microresonators are highly multimode [266] with mode-crossings that complicate analysis, suppress soliton formation [267], and add characteristic features to the spectrum [268,269]. While operation is generally in a single mode, the existence of other modes enables dispersion engineering [270–275], reduced surface-scattering loss [276], thermal nonlinearity compensation [277], and single soliton generation [278]. Researchers have also demonstrated mutually stable solitons by use of a second polarization [279], propagation direction [280,281], frequency [282,283], coupled resonator [284], or spatial mode [283,285,286]. Two solitons in distinct spatial modes can be mutually coherent with distinct repetition rates [285], as well as synchronized in time [283,286], in a process comparable to STML in fiber lasers. The incorporation of gain by coupling to a fiber laser cavity allows the formation of laser cavity solitons, which underpin microcombs with dramatically reduced threshold power and dramatically enhanced mode efficiency [287]. While a recent study examines the topological properties of multiple coupled ring cavities [288], multiple-mode (>2) operation of a microresonator has not yet been studied. Operation of simultaneous multiple modes could add channels for entanglement [289,290], telecommunications [285,291–296], and signal processing [297–305], and could increase the power per comb for improved signal to noise for imaging [306,307], ranging [308,309], and spectroscopy [310–313]. In other words, integrated frequency combs provide a simple platform for exploring the coherent nonlinear interplay of multiple modes, and the effective control of these modes will directly benefit most established applications.

I. Multimode Macroresonator Frequency Combs (Driven-Dissipative)

Beyond integrated devices, Kerr resonators have enabled important scientific and technological applications in macroscopic fiber and bulk optical platforms. The first soliton-forming Kerr resonators were demonstrated in fiber with unique benefits for photonic processing as an optical buffer [314,315]. More recently, fiber Kerr resonators have demonstrated the promise of complementing traditional mode-locked laser sources for short-pulse applications through their ability to generate high-performance ultrashort pulses in a versatile passive cavity without the fundamental wavelength and temporal limitations imposed by an active gain medium [316,317]. In parallel, bulk Kerr resonators have been established for the first time as an efficient approach to cavity enhancement, enabling record peak power enhancement for HHG [318]. While recent work shows how solitons in different polarizations of fiber can couple [319], form polarization domain walls [320], and support vectorial solitons [321], researchers have only just begun investigating nonlinear spatial mode coupling. A 2022 study examines STML in a Kerr resonator consisting of GRIN fiber, demonstrating high coherence and ultra-low timing jitter [286]. Looking forward, coherent control of multimode phenomena in macroscopic Kerr resonators has the potential for impressive processing power for telecommunications, unprecedented peak power performance, and versatility for short-pulse sources and enhancement cavities, as well beam control, combining, and spatiotemporal wave packet engineering.

J. Multimode Quantum Nonlinear Photonic Devices

Multimode quantum light can facilitate efficient scaling of photonic quantum computing protocols, which usually require high-dimensional quantum states for error correction [322–324], quantum-enhanced imaging techniques [35], or other quantum-enhanced characteristics, such as super-classical pointing stability [325]. For large-scale photonic quantum information processing systems especially, multiple DOF and high-dimensional encodings are becoming increasingly attractive and even essential [326,327]. But multimode nonlinear photonics is not only useful for quantum photonic devices, it is in many respects fundamental to it. Spontaneous nonlinear processes, like parametric downconversion and FWM, supply the majority of non-classical light. Since these processes populate all available modes that are energy- and momentum-conserving, in bulk media or multimode structures their design *requires* a multimode, spatiotemporal formalism, even with continuous-wave pumping [22,24–27,29,92–96]. While single-mode waveguides simplify design, they inevitably increase loss. In photonic integrated circuits, integrated multimode waveguides also enable efficient routes toward larger linear quantum optical circuits including mode encodings, coherent conversion schemes, and qubit or qudit states [93,326,328–331]. Quantum photonic systems based on free-space or multimode fiber [94,332–334] may, however, especially in the short-term, facilitate control of multimode quantum optical states with drastically lower loss and larger scale. Overall, quantum optics is increasingly multimode and high-dimensional both because of necessity—non-classical light generation is by default a spatiotemporal, multimode process—and opportunity—high-dimensional quantum light can be quantitatively superior for quantum computing, quantum sensing, and quantum manipulation.

K. Other Integrated Multimode Nonlinear Photonic Devices

Nonlinear photonic devices with many DOFs have been proposed to perform a wide range of functions. The noisy, analog dynamics of nonlinear optical systems makes them poorly suited to digital computations but attractive for brain-like functions, such as heuristic optimization or deep neural network inference [78,239,335–338]. A primary goal of optical neuromorphic computers is to improve energy efficiency, so the power required for typical nonlinear optics is a challenge. One solution is enhancing nonlinearities through strong spatiotemporal confinement in emerging platforms like thin-film lithium niobate [339], or in exciton-polariton [340–342] or phonon-polariton [343,344] light–matter quasiparticles, which permit spatiotemporal nonlinear optics with light levels approaching the single-photon scale [342]. Integrated multimode nonlinear photonic devices may also include systems for particle acceleration [18,19,21,104–106,110,345] or, plausibly, miniaturized electron beam sources [107,108]. In general, controlling in many dimensions the behavior of intense laser light in tightly confined integrated settings should facilitate low-cost, portable versions of lab-scale devices as well as entirely new functions, such as analog computing.

3. EMERGING METHODOLOGIES

A. Arbitrary Multimode Field Generation

The development of light sources with complete control of the spatiotemporal profile of the electromagnetic field will both enable and benefit from emerging concepts in multimode nonlinear photonics. Spatiotemporal light shaping [346–349] is rudimentary compared to spatial [1] or temporal [350,351] versions, but emerging techniques based on controlling spatio-spectral correlations (Fig. 5) [355] or coherently combining the outputs of multiple emitters (Fig. 5) [106,356,357] allow synthesis of a rich variety of localized spatiotemporal structures with separable [358] or non-separable [355] profiles. These include X-waves [359], wave packets with controllable group velocities [89,90,355,360], space-time toroidal vortices [361], and accelerating spatiotemporal packets [358,362]. The combination of a SLM and multi-plane light conversion [Figs. 5(a) and 5(b) underlie mode-based synthesis of light fields completely controlled in space, time, and polarization [363]. A major open challenge is the generation of high-power multimode light, which will be a natural tool for nonlinear spatiotemporal interactions [111,364]. High intensities will mostly preclude "carving" desired fields after the light is generated and place a premium on production of structured light from the source [Fig. 5(b)]. Appropriately designed multimode or multicore sources may exploit competition for gain in lasers [75,218,219,365] and/or scalable coherent combining schemes [106,357], and multimode propagation may even help mitigate pulse-distorting nonlinear processes.

B. Beam Self-Cleaning in Multimode Light Sources

The observation of beam-cleaning in a conservative system was quite surprising. With input power well below the critical power for self-focusing, a speckled input beam launched into passive multimode GRIN fiber evolves to a bell-shaped intensity profile as it propagates. First observed with nanosecond pulses [366], experiments with continuum generation [186] and femtosecond pulses [367] exhibited beam-cleaning soon after [30]. Much excitement for potential applications naturally ensued, and beam-cleaning was quickly exploited to enhance photoacoustic microscopy [245] and to produce a diffraction-limited beam from a multimode mode-locked fiber laser [216]. Multi-octave continua with good beam quality can be generated in multimode fiber [186], and such a continuum was employed as a stable three-color source for multiphoton microscopy and endoscopy [Fig. 5(d)] [354]. Applications of beam-cleaning with greater impact than these initial demonstrations will require deeper understanding and control of coherence as well as the dramatic spatiotemporal dynamics [368,369] that can occur in beam-cleaning processes. It is now understood that conservative beam-cleaning based on the electronic Kerr nonlinearity cannot improve the beam quality parameter M^2 [370,371], as expected from thermodynamic considerations. Nonetheless, light generated by geometric parametric instability [164] or stimulated Raman scattering [175,185,372] in multimode fibers may be dominated by the fundamental mode [177], and dissipative Kerr beam-cleaning processes may yield net benefits in fiber lasers [74,75] or amplifiers [209,210]. The judicious combination of beam-cleaning processes with dissipation offers exciting prospects for engineering light sources with enhanced performance.

C. Applications of Optical Thermodynamics to Design of Multimode Optical Systems

While emerging multimode technologies could prove to be revolutionary, they pose a new set of fundamental challenges. These stem from the extreme complexity of multimode systems, which is generally exacerbated by nonlinearities [373,374]. Nonlinear multi-wave mixing processes induce energy exchange through a multitude of possible pathways [370,375–377], which can number in the billions even in the presence of only ∼100 modes. Modeling and harnessing the response of such complex configurations will be difficult even with the advanced computing resources and techniques described above. An equilibrium optical thermodynamic theory can effectively describe the processes of energy/power exchange in conservative multimode systems with weak nonlinear interactions [378–382], and it provides intuition about multimode nonlinear evolution through maximization of photonic entropy [Fig. 5(e)]. As thermodynamic approaches are validated by experiments [371,383–385] that include beam-cleaning, it will be interesting to apply them to the design of multimode sources of the kinds discussed in Section 2. Extensions to non-equilibrium versions of optical thermodynamics are challenging conceptually, but related work [386] suggests that such approaches may become effective tools for analyzing the dissipative environments of lasers and amplifiers.

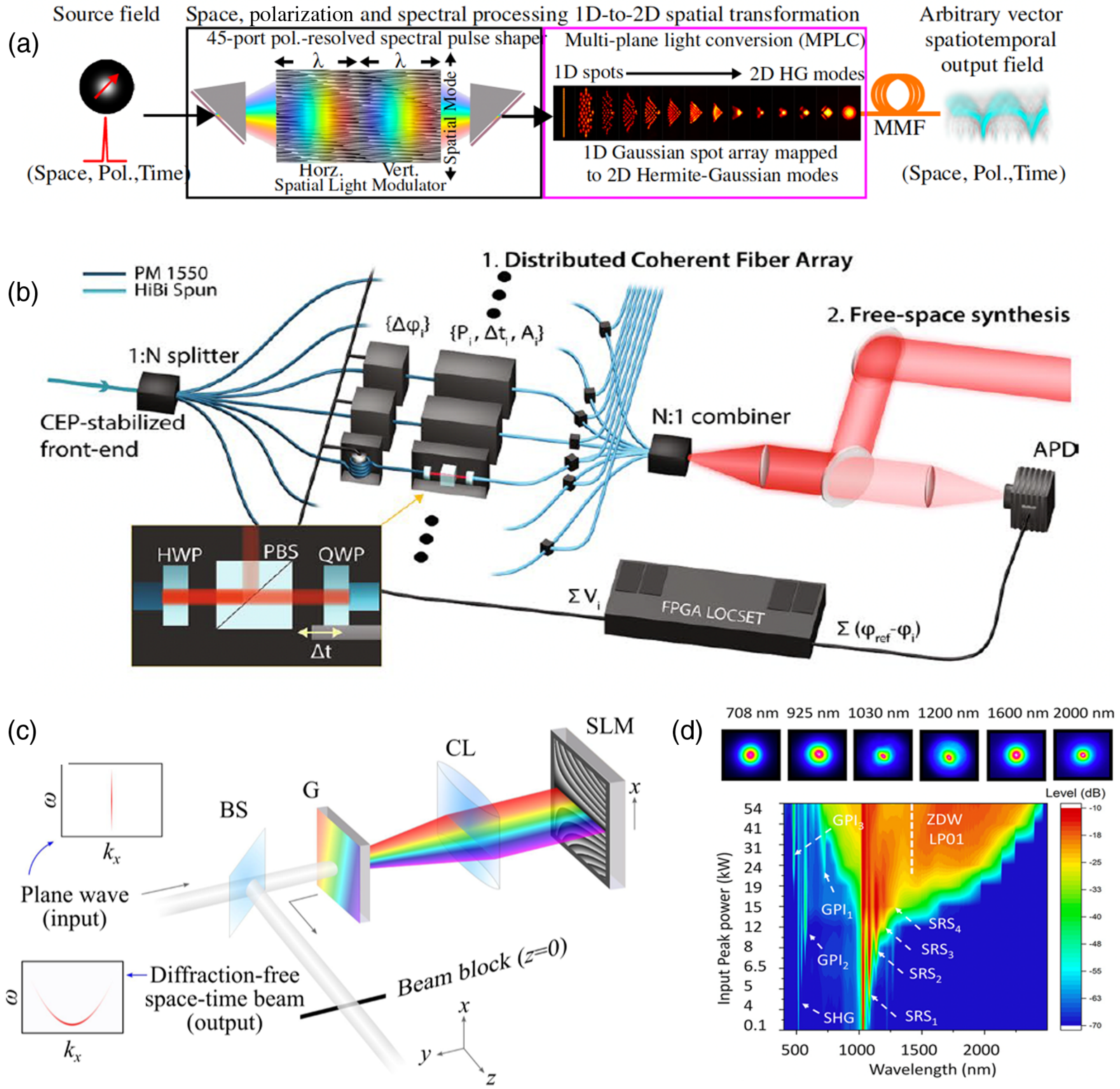


Fig. 5. Emerging methodologies relevant to multimode nonlinear photonics. (a) Schematic of a device capable of mapping an input vector spatiotemporal field onto an arbitrary vector spatiotemporal output field [352]. (b) Control of light "from the source": schematic of system for coherent spatiotemporal beam combination, a power-scalable approach for spatiotemporal arbitrary waveform generation [106]. (c) Concept and schematic apparatus for creating a class of spatiotemporally shaped fields, including space-time light-sheets [353]. (d) Output beam profiles and continuum generated in multimode fiber [354]. GPI refers to geometric parametric instability, and SRS refers to stimulated Raman scattering. Adapted from [106,352–354] with permission.

D. Applications of Data-Driven and Inverse Multimode Design

Deep learning involves computational optimization of nonlinear networks, which can often be viewed as discrete approximations of high-dimensional nonlinear dynamical systems [78,387–389]. The surprising efficacy of deep learning is now understood to result from the remarkable properties of stochastic gradient descent in high-dimensional parameter spaces [158]. These virtues of high-dimensional gradient-based optimization, along with the shrinking cost of computing [Fig. 3(c)], make computational inverse design of multimode linear and nonlinear photonic devices an increasingly feasible and exciting prospect. While so far many applications of inverse design in photonics [153,157,390] have been for relatively simple, few-mode devices, optimizations of multimode [156,391] and even multimode nonlinear photonic systems [154,238,336,338] are now appearing. To permit faster design of multimode systems with many DOFs, one family of strategies employs deep neural networks to either accelerate approximate simulations, and/or to express high-dimensional designs (often with discrete parameter spaces) in continuous lower-dimensional latent spaces [143,392–395]. Applying these techniques to multimode nonlinear devices will probably require improvements in neural network emulators of multimode nonlinear optical wave propagation, where early results are encouraging [396]. These strategies, employing relatively black-box models, may be improved on by techniques that incorporate physical insight [143,392,393]. Overall, designing nonlinear dynamics with many DOFs is challenging but increasingly feasible with computer-assisted design methodologies due to both algorithmic advances and the steady improvements in the cost of computing.

4. CONCLUSION

Enormous progress has been made in engineering nonlinear optical systems based on systems that access limited numbers of modes or DOFs, and these find widespread application. Photonic systems with enhanced or new capabilities will be possible with control of light fields and light–matter interactions with many DOFs. Understanding and design of highly multimode nonlinear optical systems certainly present major technical and conceptual challenges. However, advances in understanding of multimode linear systems, along with development of optical components with many DOFs and enhanced computing resources, create a strong foundation for addressing these challenges. Armed with these tools, we expect that future photonics engineers may design nonlinear multimode devices and instruments in ways that are presently challenging or impossible. Extrapolating from the history of nonlinear optical devices to date (Fig. 1), we expect that this new class of complex nonlinear optical devices will have major impact on optical science and will facilitate a wide range of new applications.

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